Entanglement induced by tailored environments

Markus A. Cirone, G. Massimo Palma
NEST - CNR (INFM) & Dipartimento di Scienze Fisiche ed Astronomiche,
Università di Palermo, via Archirafi 36,
I-90123 Palermo, Italy
massimo.palma@fisica.unipa.it
(Dated: October 12, 2009)

We analyze a system consisting of two spatially separated quantum objects, here modeled as two pseudo-spins, coupled with a mesoscopic environment modeled as a bosonic bath. We show that by engineering either the dispersion of the spin boson coupling or the environment dimensionality - or both - one can in principle tailor the spatial dependence of the induced entanglement on the spatial separation between the two spins. In particular we consider one, two and three dimensional reservoirs and we find that while for a two or three dimensional reservoir the induced entanglement shows an inverse power law dependence on the spin separation, the induced entanglement becomes separation independent for a one dimensional reservoir.

Keywords: Entanglement, reservoir engineering, spin boson model

Introduction

Among the various schemes that have been recently analyzed to generate entanglement between quantum subsystems, particular attention has been devoted to mechanisms based on the the interaction with a so called entanglement mediator, i.e. a third system which interacts locally with the subsystems one wants to entangle (see e.g.[1,2,3]). Interestingly enough the role of entanglement mediator can be played by a large quantum reservoir. For example it has been shown that two atoms or two quantum dots interacting with the quantized electromagnetic field become steadily entangled even at finite temperature[4]. At the same time it has been recently shown how cold atoms can be used to engineer, with a high degree of flexibility, systems described by spin boson Hamiltonians[5]. This opens the possibility to implement new schemes in which the desired entanglement between two microscopic quantum objects can be controlled by a suitable design of a mesoscopic reservoir. In the present paper we will briefly analyze how, by engineering either the dispersion of the spin coupling or the dimensionality of the reservoir (or both) one can control the spatial dependence of the entanglement induced by a bosonic bath on two spatially separated quantum objects.

The model

To illustrate the idea consider a system consisting of two two-level systems, labeled α, β located at positions $\mathbf{r}_\alpha, \mathbf{r}_\beta$ respectively and interacting with a bosonic scalar quantum field. We describe such a system by the following spin boson Hamiltonian

$$H = H_0 + H_i \tag{1}$$

with

$$H_0 = \frac{\omega_0}{2}\left(\sigma_z^\alpha + \sigma_z^\beta\right) + \sum_k \omega_k a_k^\dagger a_k \tag{2}$$

$$H_i = \sum_{i=\alpha\beta} \sum_k \left[\left(\lambda_k(\mathbf{r}_i) a_k + \lambda_k^*(\mathbf{r}_i) a_k^\dagger\right)\left(\sigma_+^i + \sigma_-^i\right)\right] \tag{3}$$

where ω_0 is the energy difference between ground $|g\rangle$ and excited state $|e\rangle$ (we assume $\hbar = 1$), $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$ are pseudospin operators and a_k and $a_k^\dagger$ denote bosonic annihilation and creation operators respectively, of environment quanta of energy ω_k. The $\lambda_k(\mathbf{r}_i)$ are position and frequency dependent coupling constants between the atom j and field mode k and contain all the information on the spatial dependence of the quantized field modes. We will leave their specific functional form unspecified as long as possible in order to simplify the calculations and to obtain expressions which are as general as possible. Note incidentally that the interaction Hamiltonian H_i differs from the one examined in[4].

Straightforward time independent perturbation theory[6], leads to the following expressions for the the perturbed (dressed) ground state $|\Psi\rangle$ of the overall system system and for the interaction energy Δ

$$
\begin{aligned}
|\Psi\rangle &= c_{gg}|g_\alpha g_\beta\rangle|0_k\rangle + \frac{Q}{E_0 - H_0}\left(H_i - \Delta\right)|\Psi\rangle && (4)\\
\Delta &= \langle g_\alpha g_\beta|H_i|\Psi\rangle && (5)
\end{aligned}
$$

where $|g_\alpha g_\beta\rangle|0_k\rangle$ is the unperturbed ground state with energy $E_0 = -\omega_0$ and $Q = \mathcal{I} - |g_\alpha g_\beta\rangle\langle g_\alpha g_\beta|$ is a projector acting on the atomic degrees of freedom. The state equation (4) is in closed form and can be expanded at the desired order of approximation. The coefficient c_{gg} is chosen in order to normalize $|\Psi\rangle$.

The normalized, "dressed" ground state of the system can be written at second order of approximation in the following compact form

$$
\begin{aligned}
|\Psi^{(2)}\rangle &= c_{gg}|g_\alpha g_\beta\rangle|0_k\rangle + c_{ee}|e_\alpha e_\beta\rangle|0_k\rangle + \sum_k c_{eg,k}|e_\alpha g_\beta\rangle|1_k\rangle + \sum_k c_{ge,k}|g_\alpha e_\beta\rangle|1_k\rangle \\
&+ \frac{1}{2}\sum_{kk'} c_{gg,kk'}|g_\alpha g_\beta\rangle|1_k 1_{k'}\rangle + \frac{1}{2}\sum_{kk'} c_{ee,kk'}|e_\alpha e_\beta\rangle|1_k 1_{k'}\rangle && (6)
\end{aligned}
$$

The ground state entanglement

To quantify the entanglement induced by the reservoir between the two pseudospins we will use the two-tangle, i.e. the square of the concurrence. We remind that given the density operator $\rho_{\alpha\beta}$ of a bipartite system of two qubits, the 2- tangle $\tau_{\alpha|\beta}$ is defined as

$$
\tau_{\alpha|\beta}(\rho) = [\max\{0, \xi_1 - \xi_2 - \xi_3 - \xi_4\}]^2, \tag{7}
$$

where $\{\xi_i\}$ $(i = 1, .., 4)$ are the square roots of the eigenvalues (in non-increasing order) of the non-Hermitian operator $\bar{\rho} = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, σ_y is the y-Pauli operator and ρ^* is the complex conjugate of ρ, in the eigenbasis of $\sigma_z \otimes \sigma_z$ operator. For states of the form like (6), the 2-tangle takes the following simple form[8]:

$$
\begin{aligned}
\tau_{\alpha|\beta} &= 4\,|\langle e_\alpha e_\beta 0_k|\Psi^{(2)}|^2 = 4\,|\langle e_\alpha e_\beta 0_k|\frac{Q}{E_0 - H_0}H_i\frac{Q}{E_0 - H_0}H_i|g_\alpha g_\beta 0_k\rangle|^2 \\
&= \frac{4}{\omega_0^2}\left|\sum_k \frac{\lambda_k(\mathbf{r}_\alpha)\lambda_k^*(\mathbf{r}_\beta)}{\omega_0 + \omega_k}\right|^2 && (8)
\end{aligned}
$$

In order to extract information on the dependence of the induced entanglement on the spatial separation between the two spins we must now specify the functional form of the coupling constants $\lambda_k(\mathbf{r}_i)$. As far as the mode structure is concerned, we assume the usual plane-wave expansion

$$
\lambda_k(\mathbf{r}_i) \rightarrow \lambda_{\mathbf{k}}(\mathbf{r}_i) = \epsilon_{\mathbf{k}}\frac{e^{i\mathbf{k}\cdot\mathbf{r}_i}}{\sqrt{V}} \tag{9}
$$

The coupling factor $\epsilon_{\mathbf{k}}$ contains spin parameters which we do not need to specify for our purposes. We will simply assume that it depends on the field mode frequency spectrum via a power law, i.e.

$$
\epsilon_{\mathbf{k}} \propto \omega_{\mathbf{k}}^{\nu} \tag{10}
$$

where $\mathbf{k}$ is the momentum of the oscillators. When $V \rightarrow \infty$ the sum over momenta $\mathbf{k}$ can be replaced by an integral

$$
\sum_k \frac{\lambda_k(\mathbf{r}_\alpha)\lambda_k^*(\mathbf{r}_\beta)}{\omega_0 + \omega_k} \rightarrow \frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{\epsilon_{\mathbf{k}}^A \epsilon_{\mathbf{k}}^B e^{i\mathbf{k}\cdot\mathbf{r}}}{\omega_0 + \omega_{\mathbf{k}}} \propto \int dk\, k^{d-1}\frac{k^{2\nu}f(kr)}{k_0 + k} \tag{11}
$$

where $\mathbf{r} = \mathbf{r}_A - \mathbf{r}_B$, d is the spatial dimension and a linear dispersion relation between $\omega_{\mathbf{k}}$ and $k = |\mathbf{k}|$ has been assumed. The function $f(kr)$ comes from the angular integration $\int d\Omega_{\mathbf{k}}$ and depends both on the spatial dimension d and on the dispersion law of the coupling. In the atom-radiation interaction the coupling has vectorial character and an r^{-6} scaling of the tangle is found[9]. Here for the sake of simplicity we assume a scalar coupling. We examine now in some detail the dependence of energy and tangle on the spatial dimension d and on the value of ν.

$d = 3$

In three dimensions the angular integration in (11) gives

$$f(kr) \;=\; \int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = \int d\phi \int d\theta \sin\theta e^{ikr\cos\theta} = 4\pi \frac{\sin kr}{kr}, \tag{12}$$

Assuming $\nu = 1/2$, as in the case of the coupling with the electromagnetic field, we obtain

$$\sum_j \frac{g_j(\mathbf{r}_A) g_j^*(\mathbf{r}_B)}{\omega_0 + \omega_j} \;\propto\; \frac{1}{r} \int dk\, k^2 \frac{\sin kr}{k_0 + k} \tag{13}$$

The integral present in (13) formally diverges. The introduction of a cutoff momentum k_c (always present in real physical systems) gives

$$\begin{aligned}\int_0^{k_c} dk \frac{k^2 \sin kr}{k + k_0} \;&=\; \frac{1}{r^2} \left\{ (k_0 - k_c)\, r \cos k_c r + \sin k_c r + \right. \\ &+\; k_0 r \left[-1 + k_0 r \sin k_0 r \left(\mathrm{Ci}(k_0 r) - \mathrm{Ci}(k_0 r + k_c r) \right) \; + k_0 r \cos k_0 r \left(-\mathrm{Si}(k_0 r) + \mathrm{Si}(k_0 r + k_c r) \right) \right] \left. \right\}\end{aligned} \tag{14}$$

Neglecting the unphysical fast oscillating terms $\cos k_c r$ and $\sin k_c r$ and and keeping the leading terms in $k_0 r \ll 1$, the tangle scales as r^{-4} with the distance. Clearly, different couplings lead to different scaling laws. For instance, for flat, i.e. frequency independent coupling , $\nu = 0$ and the tangle scales as r^{-2}.

$d = 2$

In two dimensions we have

$$f(kr) \;=\; \int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = \int d\phi e^{ikr\cos\phi} = 2\pi J_0(kr), \tag{15}$$

where J_0 denotes the Bessel function of order 0. Thus

$$\sum_j \frac{g_j(\mathbf{r}_A) g_j^*(\mathbf{r}_B)}{\omega_0 + \omega_j} \;\propto\; \int dk\, k^{1+2\nu} \frac{J_0(kr)}{k_0 + k} \tag{16}$$

Numerical calculations show that this quantity scales approximately as r^{-1} for $\nu = 0$ and $\nu = 1/2$, and as r^{-3} for $\nu = 1$.

$d = 1$

For a one-dimensional environment we get

$$\sum_j \frac{g_j(\mathbf{r}_A) g_j^*(\mathbf{r}_B)}{\omega_0 + \omega_j} \;\propto\; \int_{-\infty}^{\infty} dk\, k^{2\nu} \frac{e^{ikr}}{k_0 + k} \tag{17}$$

Assuming $2\nu = n$, where n is an integer, we get

$$\begin{aligned}\sum_j \frac{g_j(\mathbf{r}_A) g_j^*(\mathbf{r}_B)}{\omega_0 + \omega_j} \;&\propto\; \int_{-\infty}^{\infty} dk\, k^n \frac{e^{ikr}}{k_0 + k} = \int_0^{\infty} dk\, k^n \frac{e^{ikr}}{k_0 + k} - (-1)^n \int_0^{\infty} dk\, k^n \frac{e^{-ikr}}{k - k_0} \\ &= \frac{1}{i^n} \frac{\partial}{\partial r^n} \left[\int_0^{\infty} dk\, k^n \frac{e^{ikr}}{k_0 + k} - \int_0^{\infty} dk\, k^n \frac{e^{-ikr}}{k - k_0} \right] = (-1)^n\, i\pi k_0^n e^{-ik_0 r}\end{aligned} \tag{18}$$

i.e. the tangle turns out to be independent of the spin separation r.

conclusions

In summary we have shown that the interaction of two spatially separated pseudospins with a common environment leads to an entanglement whose scaling on the spin spatial separation depends on the dimensionality of the environment and on how the coupling constants scale with respect to the frequency of the reservoir modes. This result holds for the class of physical systems described by a spin-boson Hamiltonian of the form (3). In particular, for a three and two dimensional environment we have found a dependence which scales as r^{-n}, with different values of n depending on the dispersion of the coupling constants. On the contrary, in one dimension the tangle does not depend on the distance between the two spins.

Note that our analysis has been carried on by a perturbative expansion of the ground "dressed" state of the overall atoms field system. In other words we have studied the equilibrium state of the overall system. Although a full time dependent analysis of the atomic entanglement during the process of approach to equilibrium is out of the scope of the present articles we would like to point out the difference between our approach and some of the existing literature. Some aspects of the time-dependent entanglement which builds up in the collective decay of two subsystems interacting with a common bath has been analyzed in[10,11]. In particular they both study the time dependent as well as the asymptotic entanglement relation with the collective damping rate. In both studies however, consistently with the Wigner Veisskopf approach to the decay process, the ground state is not dressed. Here instead we have focussed our attention on the role of the bath as an entanglement mediator in the ground state. Of course, in the presence of collective interactions with a reservoir, there may exist subradiant states which are effectively decoupled from the environment. The entanglement which would characterize such state is out of the scope of this manuscript and cannot be analyzed with the techniques used above.

Acknowledgments

We acknowledge support from the PRIN 2006 "Quantum noise in mesoscopic systems" and from EUROTECH S.p.A.

---

[1] F. Plastina and G. Falci, Phys. Rev. B 67, 224514 (**2003**).
[2] A. Blais, A. van den Brink, and A. M. Zagoskin Phys. Rev. Lett. 90, 127901 (**2003**)
[3] J. Majer, J. M. Chow, J. M. Gambetta, Jens Koch, B. R. Johnson, J. A. Schreier, L. Frunzio, D. I. Schuster, A. A. Houck, A. Wallraff, A. Blais, M. H. Devoret, S. M. Girvin and R. J. Schoelkopf Nature 449, 443, (**2007**)
[4] D. Braun, Phys. Rev. Lett. 89, 277901 (**2002**); D. Braun, Phys. Rev. A 72, 062324 (**2005**).
[5] D. Jaksch and P. Zoller, Annals of Physics 315, 52, (**2005**)
[6] J. J. Sakurai, Modern Quantum Mechanics, Addison-Wesley (Reading, Massachusetts) (**1994**).
[7] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (**1998**).
[8] M. A. Cirone, G. Compagno, G. M. Palma, R. Passante, F. Persico, Europhys. Lett. 78, 30003 (**2007**).
[9] D. P. Craig, T. Thirunamachandran, Molecular Quantum Electrodynamics, Dover (New York) (**1998**);
[10] F. Benatti, R. Floreanini, and M. Piani, Phys. Rev. Lett. 91, 070402 (**2003**), F. Benatti and R. Floreanini, Phys. Rev. A 70, 012112 (**2004**)
[11] Z. Ficek and R. Tanas Physics Reports 372, 369, (**2002**)